帽子的顏色

三個男孩子坐成一行，面向同一方向，最後的一個能看到前面二人所戴的帽子是甚麼顏色，中間的只看到前面的帽子，最前面的則看不到任何一人。

三人先閉上眼睛，老師在兩頂紅帽及三頂白帽中抽出三頂，分別為他們戴上。若後面兩個同學也答不出自己帽子的顏色，坐最前面的同學的帽子應是甚麼顏色呢？

1.1 The Multiplication Principle of Counting *P.2*

If a first operation can be performed in n_1 ways, a second operation in n_2 ways, a third operation in n_3 ways, and so forth, then the sequence of ***k*** operations can be performed in $n_1 \, n_2 \ldots n_k$ ways.

Exercise 1.1 *P.5*

2. **A child has 6 sweaters and 7 pairs of trousers. How many different outfits can he/she wear?**
The number of different outfits =

5. **A student must take a social science subject and a language subject. The social science subjects available are *Economics*, *Geography* and *Commerce*. The language subjects available are *English*, *Chinese*, *Japanese*, *German* and *French*. How many ways can the student arrange his/her study program?**
The number of ways =

7. **A computer shop has 5 types of monitors, 4 CPU systems and 7 printers available. How many different computer systems consisting of a monitor, a CPU system and a printer can be formed?**
The number of different computer systems =

9. **A four-course dinner consists of a soup, a main dish, a dessert and a drink. If one can select from 4 different soups, 5 main dishes, 3 desserts and 6 drinks, how many dinner choices are possible?**
The number of dinner choices =

11. **There is a 6-digit Personal Identification Number (PIN) encoded in each bank card for security reasons. Find the number of possible PINs**

(a) **with repeated digits allowed,**
The number of possible PINs =

(b) **with no repeated digits.**
The number of possible PINs =

18. **A vehicle licence plate consists of 2 letters followed by 4 digits.**

(a) **How many different plates are possible with no restriction?**
The number of different plates =

(b) **How many different plates are possible if the letters O and I are excluded?**
The number of different plates =

(c) **How many different plates are possible if the letters O and I are excluded and at least one of the digits is 8?**
If the digit 8 is excluded, the number of different plates =
$\therefore$ The required number =

1.2 Permutations *P.7*

The factorial notation *P.7*

$n! = n(n-1)(n-2)...(3)(2)(1)$
$0! = 1$

Exercise 1.2 *P.13*

Evaluate the following expressions.

3. $\frac{9!}{4!5!}$ [126] **4.** $\frac{11!}{3!8!}$ [165] **6.** P_4^7 [840] **7.** $\frac{(n+6)!}{(n+4)!}$ [(n + 6)(n + 5)]

Number of permutations of n distinct objects is $\boldsymbol{n!}$.

12. How many ways are there to arrange 7 balls of different colours in a line?

The number of ways =

The Permutation Symbol *P.8*

Number of permutations of n distinct objects taken r at a time, without repetitions, is $P_r^n = \frac{n!}{(n-r)!}$ or $P_r^n = n(n-1)(n-2).....(n-r+1)$

15. From a collection of 10 hottest songs, a list ranking the top 4 must be made. How many ways are there to make such a list?

The number of ways =

16. How many different arrangements of 3 letters can be formed from the 26 letters A to Z if repeated letters are not allowed?

The number of ways =

Permutation of n objects, not all distinct *P.9*

Number of permutations of n distinct objects taken all together, in which there are p alike of one kind, q alike of a second kind, r alike of a third kind and so on, is $\frac{n!}{p!q!r!}$ *where* $\boldsymbol{n = p + q + r}$

18. How many distinct permutations can be made from the letters of the word *"EXCELLENT"*?

Because there are 3 *E*'s and 2 *L*'s, the number of permutations =

Circular Permutations *P.11*

The number of circular permutations of n distinct objects taken all together is $\boldsymbol{(n-1)!}$ when clockwise order is distinguished.

If the clockwise and counter-clockwise order are not distinguished, the number of ways is $\frac{1}{2}\boldsymbol{(n-1)!}$.

22. In how many ways can 6 different joysticks be arranged in a circle for display?

The number of ways =

26. 5 boys and 4 girls are arranged to sit in a row. How many possible arrangements are there if

(a) there is no restriction?

The number of ways =

(b) a particular boy must sit in the leftmost?

The number of ways =

(c) the 5 boys are on one side?

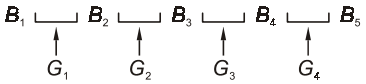
The number of permutations of 5 boys in a line =

The number of permutations of 4 girls in a line =

Since the boys may be on the left or right of the girls,,

Req'd no. of ways =

(d) the boys and girls must alternate?



For each arrangement of the 5 boys, we can associate it with an arrangement of the 4 girls and place the girls in between the boys as shown in the figure.

∴ The required number of ways =

27. 7 distinct flags are hoisted in a post. Find the number of ways of arranging them if

(a) 4 of the flags must be together,

The number of ways of arranging the group of 4 flags and the remaining 3 flags =

For each such arrangement, the group of 4 flags can be arranged among themselves in 4! ways.

∴ The required number of ways =

(b) 2 of the flags must be separated.

On the same line of arguments as (a), if 2 flags must always be together,

the number of ways of hoisting =

The number of ways of hoisting 7 flags without any condition =

∴ If 2 flags must always be separated,

the number of ways of hoisting =

29. Susan decorates her Christmas tree using a series of light bulbs. She has 5 red, 4 yellow and 2 blue bulbs available.

(a) (i) If she uses all the light bulbs, how many different arrangements can she make?

The required number of arrangements =

(ii) In (i), how many of the arrangements will the two blue bulbs be separated?

The number of arrangements with the two blue bulbs together =

∴ The number of arrangements with the two blue bulbs separated =

(b) If she only uses 10 of the bulbs, how many different arrangements can she make?

If she uses 5 red, 4 yellow and 1 blue bulbs,

the number of ways =

If she uses 5 red, 3 yellow and 2 blue bulbs,

the number of ways =

If she uses 4 red, 4 yellow and 2 blue bulbs,

the number of ways =

∴ The total number of different arrangements =

1.3 Combinations *P.15*

A B C (3 choose 2)	**Permutation 排列**	**Combination 組合**
	AB BA AC CA BC CB	AB AC BC
	$6 = P_2^3$	$3 = C_2^3$

The Combination Symbol *P.15*

Number of combinations of n distinct objects taken r at a time is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Exercise 1.3 *P.19*

Evaluate the following.

2. C_3^7 [35] **6.** C_{17}^{19} [171]

11. In how many ways can a committee of 3 principals be selected from a group of 10 principals?

The number of ways =

12. There are 25 students in the Science Club. In how many ways can 4 officers be selected?

The number of ways =

Useful relations for Combinations *P.16*

a. $C_r^n = C_{n-r}^n$

b. $C_r^{n+1} = C_{r-1}^n + C_r^n$

17. A relay team of 4 persons is selected from a group of 9 runners. How many different teams can be formed if

(a) an outstanding runner must be included in the team?

We form the team by selecting 3 persons from the remaining 8 runners.

∴ The number of different teams =

(b) a wounded runner must be excluded from the team?

We have to select 4 persons from the remaining 8 runners.

∴ The number of different teams =

20. In how many ways can a group of 5 printers be selected from 6 inkjet and 9 laser printers if the group must contain

(a) exactly 3 laser printers?

The number of ways that include exactly 3 laser printers =

(b) at least 3 laser printers?

The number of ways that include exactly 4 laser printers =

The number of ways that include exactly 5 laser printers =

∴ The number of ways that include at least 3 laser printers =

21. A box contains 4 green apples and 8 red apples. In how many ways can a child pick 3 of these apples and receive

(a) exactly 2 green apples?

The number of ways that contain exactly 2 green apples =

(b) at most 1 green apple?

The number of ways that contain no green apple =

The number of ways that contain exactly 1 green apple =

∴ The number of ways that contain at most 1 green apple =

25. A human resource manager has to locate 15 clerks into 3 offices that require 7, 5 and 3 clerks respectively. In how many ways can the 3 groups of clerks be chosen?

The number of ways =

27. A sample of 5 cars is selected for a destructive test from a group containing 7 small, 4 medium and 6 large cars. How many samples are possible

(a) with no restriction?

The number of possible samples with no restriction =

(b) with 2 small, 2 medium and 1 large cars?

The number of possible samples with 2 small, 2 medium and 1 large cars =

(c) with at least 1 small car?

The number of possible samples with no small cars =

∴ The number of possible samples with at least 1 small car =

29. A student has 15 different books. In how many ways can he/she arrange 12 of them on a shelf if

(a) there is no restriction?

The number of ways with no restriction =

(b) a particular book must be on the shelf?

If a particular book must be on the shelf, we have to select 11 books from the 14 remaining ones.

∴ The number of ways =

(c) 2 particular books must be on the shelf and placed together?

The number of ways =

30.# 8 out of 12 scouts are selected to stand half on each side of a hall entrance. Find the number of ways in which this may be done if

(a) there is no restriction,

The number of ways of selecting 4 scouts standing on the left and arranging them in a line =

The number of ways of selecting 4 scouts from the 8 remaining ones standing on the right and arranging them in a line =

∴ The required number of ways =

(b) the 2 tallest scouts must stand at the end of each side.

The number of ways of selecting the left side =

The number of ways of selecting the right side =

∴ The required number of ways =

Revision Exercise 1 *P.23*

11. The think-tank of a firm comprises 3 representatives from each of 4 departments: production, marketing, finance and personnel. A task group of 4 persons is randomly selected from the think-tank. Find the number of different task groups that can be formed if

(a) no person is selected from the personnel department,

If no person is selected from the personnel department,
the number of ways =

(b) one person is selected from each department,

If one person is selected from each department,
the number of ways =

(c) one person is selected from the production department, two persons from the marketing department and one from either of the other two departments,

The required number of ways =

(d) the finance manager, who is a representative of finance department in the think-tank, must be on the task group.

The required number of ways =

12. A flock of 6 birds is to be chosen from 10 blue and 5 yellow birds in a cage. Find the number of ways this flock may be chosen if

(a) a particular bird must be in the flock,

The required number of ways =

(b) (i) the flock must contain at least 3 blue birds and at least two yellow birds,

The number of ways with 3 blue and 3 yellow birds =
The number of ways with 4 blue and 2 yellow birds =
$\therefore$ The required number of ways =

(ii) and if, in addition to (i), two particular blue birds cannot be placed together in the flock.

The required number of ways =

Appendix

Distribution in group分組組合

I. Specified groups (or have group name)

1. The no. of ways to divide n unlike things into 3 ***unequal groups*** containing respectively p, q, r things is $C_p^n \times C_q^{n-p} \times C_r^{n-p-q}$ (where $n = p+q+r$).
2. The no. of ways to divide n unlike things into 3 ***equal groups*** is $C_p^n \times C_p^{n-p} \times C_p^{n-2p}$ (where $n = 3p$).

II. Without specified groups (or no group name)

1. The no. of ways to divide n unlike things into 3 ***unequal groups*** containing respectively p, q, r things is $C_p^n \times C_q^{n-p} \times C_r^{n-p-q}$ (where $n = p+q+r$).
2. The no. of ways to divide n unlike things into 3 ***equal groups*** is $C_p^n \times C_p^{n-p} \times C_p^{n-2p} \times \frac{1}{3!}$ (where $n = 3p$).

Example 1

How many ways, if we divide 4 different things into 2 groups A & B,

(a) group A has 1 thing and group B has 3 things.

(b) each group has 2 things.

Solution

(a) no. of ways $= C_1^4 \times C_3^3 = 4$

(b) no. of ways $= C_2^4 \times C_2^2 = 6$

Example 2

How many ways, if we divide 4 different things into 2 groups,

(a) one group has 1 thing and the other group has 3 things.

(b) each group has 2 things.

Solution

(a) no. of ways $= C_1^4 \times C_3^3$ or

$= C_3^4 \times C_1^1 = 4$

(b) no. of ways $= C_2^4 \times C_2^2 \times \frac{1}{2!} = 3$

Exercises

How many ways, if we divide 9 differents things into 3 groups,

(a) 1 group has 2 things, 1 group has 3 things, 1 group has 4 things. [***1260***]

(b) 1 group has 1 thing, 2 groups each has 4 things. [***315***]

(c) each group has 3 things. [***280***]

III. Proof

Let the 4 things be a, b, c, d

1(a) The groups are as following:

Group	*A*	B
	a	*bcd*
	b	*acd*
	c	*abd*
	d	*abc*

∴ the total no. of ways = 4

1(b) The groups are as following:

Group	*A*	B
	ab	*cd*
	ac	*bd*
	ad	*bc*
	bc	*ad*
	bd	*ac*
	cd	*ab*

∴ the total no. of ways = 6

2(a) The groups are as following:

a	*bcd*
b	*acd*
c	*abd*
d	*abc*

∴ the total no. of ways = 4

2(b) The groups are as following:

ab	*cd*
ac	*bd*
ad	*bc*
bc	*ad*
bd	*ac*
cd	*ab*

The last 3 groups are the same as first 3 groups,

∴ so the total no. of ways = 3