2.1 The Binomial Theorem

$(a+x) =$

$(a+x)^2 =$

$(a+x)^3 =$

$(a+x)^4 =$

$(a+x)^5 =$

$(a+x)^6 =$

(A) By considering the coefficients, we can have the following patterns:

					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

_____ _____ _____ _____ _____ _____ _____ _____

1. the pattern is called ***Pascal's Triangle***.
2. there are *n + 1* terms in the *n th* row.
3. each row begins and ends with a 1.
4. the triangle is symmetric.
5. *every coeff. of a row can be obtained by adding the two coeff. on its left and right in the row above it.*

(B) By considering the terms:

Sum of the powers of 2 elements of a term is equal to n.

Example

Expand

1. $(ax+y)^5$

   =

2. $(x-by)^4$

   =

Remarks (*Terminology***)**

(ascending / descending powers of x as far as the term in x^3, constant term or term independent of x, …)

(C) Relation between Binomial coefficients & Combination

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

C_r^n	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	
$n = 1$								
$n = 2$								
$n = 3$								
$n = 4$								
$n = 5$								
$n = 6$								

Remarks : 1. n and r must be **positive integers**.

2. n ≥ r.

3.* ${}_nC_r \ or \ C_r^n = \dfrac{n!}{r!(n-r)!}$.

*** — Appendix 1**

∴ **Pascal's Triangle with Combinational coefficients :**

$C_0^1 \quad C_1^1$

$C_0^2 \quad C_1^2 \quad C_2^2$

$C_0^3 \quad C_1^3 \quad C_2^3 \quad C_3^3$

$C_0^4 \quad C_1^4 \quad C_2^4 \quad C_3^4 \quad C_4^4$

$C_0^5 \quad C_1^5 \quad C_2^5 \quad C_3^5 \quad C_4^5 \quad C_5^5$

_____ _____ _____ _____ _____ _____ _____

The Binomial Theorem (for positive integral index)

$$(a+x)^n = C_0^n a^n + C_1^n a^{n-1}x + C_2^n a^{n-2}x^2 + \cdots\cdots + C_r^n a^{n-r}x^r + \cdots\cdots + C_n^n x^n$$

$$= \sum_{r=0}^{n} \underline{C_r^n a^{n-r} x^r} \quad \leftarrow \textit{general term}$$

where 1. $C_r^n = \dfrac{n!}{r!(n-r)!}$

2.* $C_r^n = C_{n-r}^n$

3.* $C_r^{n+1} = C_{r-1}^n + C_r^n$

*** — Appendix 2**

Example

1. Find the 4th term in the expansion of $(4x+5y)^{10}$ in descending powers of x. [***245760000x^7y^3***]

2. In the expansion of $\left(x^2 - \dfrac{2}{x}\right)^{12}$, find

   (a) the term independent of x, [***126720***]

   (b) the coefficient of x^9. [***-25344***]

2.3 The Binomial Series

(A) Consider the infinite geometric series

(a)* $1+x+x^2+x^3+\cdots =$

(b)* $1-x+x^2-x^3+\cdots =$

Restriction : $-1 < x < 1$ or $|x| < 1$

*** — Appendix 3**

(B) The General Binomial Theorem

$(1+x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots\cdots + C_r^n x^r + \cdots\cdots + C_n^n x^n$

If n is not a positive integer:

$$\Rightarrow C_0^n + C_1^n x + C_2^n x^2 + \cdots\cdots + C_r^n x^r + \cdots\cdots + C_n^n x^n + \cdots\cdots \quad \textit{(infinity series)}$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots\cdots + \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{r!}x^r + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r \qquad \textit{general term}$$

∴ When $n=-1$

$(1+x)^{-1} =$

(C) Restriction

$-1 < x < 1$ and n is a rational number $\Rightarrow$ ***Convergent Series*** 收斂級數

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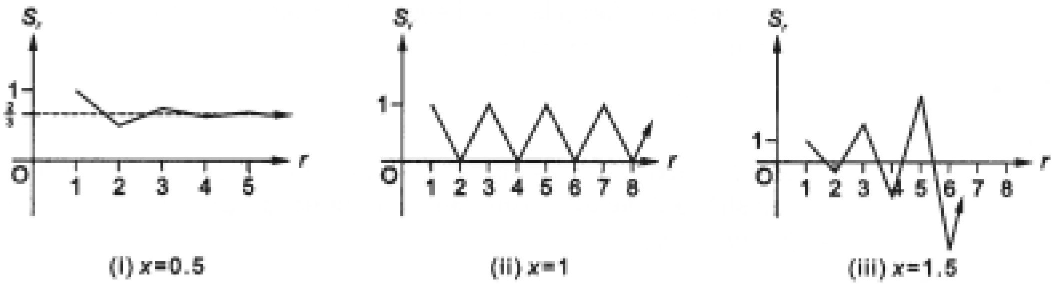

Figure 2.2 $S_r = 1 - x + x^2 - \ldots + (-1)^{r+1}x^r$

Example

1. Find the expansion of $(1-x)^{-1}$ for $|x|<1$.

2. For $|x|<1$, find the coefficient of x^3 in the expansion of $(1+x)^{\frac{1}{2}}$.

3. (a) Find the expansions of $(1+2x)^{\frac{1}{3}}$ in ascending powers of x as far as the term containing x^3.
   (b) State the range of values of x for which the expansion is valid.

4. **HKASL 1996 Q8**(a)(i),(b)(i)-(iii)

5. ***Revision Exercise 2*** **Q20**

Summary

(A) If n is positive integer $\Rightarrow$ ***Binomial Thm*** (finite no. of terms)

$$(1+x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots\cdots + C_r^n x^r + \cdots\cdots + C_n^n x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots\cdots + \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + \cdots\cdots + x^n$$

(B) If n is not positive integer $\Rightarrow$ ***Binomial Series*** (infinite no. of terms)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + \cdots$$

2.2 Summation Notation

$$\boxed{\sum_{i=1}^{n} x_i} = x_1 + x_2 + x_3 + \cdots + x_n$$

Summation Rules

If *a* and *b* are constants, then

(a) $\sum_{i=1}^{n} (ax_i \pm by_i) = a\sum_{i=1}^{n} x_i \pm b\sum_{i=1}^{n} y_i$

(b) $\sum_{i=1}^{n} (ax_i \pm b) = a\sum_{i=1}^{n} x_i \pm nb$

(c) $\sum_{i=1}^{n} (x_i \pm y_i)^2 = \sum_{i=1}^{n} x_i{}^2 \pm 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i{}^2$

Exercises 2.2 Q3, 8, 19

Appendix 1

Use of Calculator(Casio) to evaluate *Factorials*

Chapter 2 Function Keys

Factorial

Five women work in the plant; of these groups, how many would include 4 women?

$$\frac{5!}{4!(5-4)!}\times\frac{8!}{1!(8-1)!}=\frac{5!}{4!}\times\frac{8!}{7!}$$

Key in:	*Answer:*
5 [x!] [÷] 4 [x!] [×] 8 [x!] [÷] 7 [x!] [=]	**40**

How many groupings will include 5 women?

The above calculation may be carried out directly on calculators that have combination (nCr) and permutation (nPr) keys.

Key in:	*Answer:*
13 [nCr] 5 [=]	**1287**
5 [nCr] 4 [×] 8 [nCr] 1 [=]	**40**

The possible arrangements that can be made from a particular group is called the number of permutations which is defined as:

$$\frac{n!}{(n-r)!}$$

where:
n = number of objects
r = the number taken at one time

The "Trifecta" in horse racing means selecting and betting on the first, second and third finishers within a race. If there are 8 entrants in a race, what are the odds of winning?

Key in:	*Answer:*
8 [nPr] 3 [=]	**336 (1 out of 336)**

The Trifecta "Box" means selecting the three winners regardless of their order of finish. Selection of a grouping without a specified order is known as a combination. The number of combinations will always be smaller than the number of permutations for the same values.

Key in:	*Answer:*
8 [nCr] 3 [=]	**56**

PART I: Basics

Chapter 2 Function Keys

Factorial

The factorial is important in several areas of mathematics including statistics. The factorial of a number is that number multiplied by itself less one and then by that number less one until one is reached (x is a whole number only).

$$x! = x\times(x-1)\times\ldots\times4\times3\times2\times1$$
$$5! = 5\times4\times3\times2\times1 = 120$$
$$1! = 0! = 1$$

The factorial key is an operator.

Calculate: 10! ÷ 8!; 10! − 9!; 69!; 70!

Key in:	*Answer:*
10 [x!] [÷] 8 [x!] [=]	**90**
10 [x!] [−] 9 [x!] [=]	**3,265,920**
69 [x!]	**1.711224523×10^{98}**
70 [x!]	**E**

Note: *The value of the factorial for numbers greater than 69 exceeds the capacity of the calculator and will result in an overflow error.*

Five employees will be evaluated as they work in groups of two. How many different groups are possible?

$$\frac{5!}{(5-2)!}=\frac{5!}{3!}$$

Key in:	*Answer:*
5 [x!] [÷] 3 [x!] [=]	**20**

The five employees will be drawn from a total of 13 working in a plant. How many combinations of five are possible.

$$\frac{13!}{5!(13-5)!}=\frac{13!}{5!\times8!}$$

Key in:	*Answer:*
13 [x!] [÷] [(] 5 [x!] [×] 8 [x!] [=]	**1287**

PART I: Basics

Appendix 2

Proof:

2.
$$\begin{aligned}C^n_{n-r} &= \frac{n!}{(n-r)![n-(n-r)]!}\\ &= \frac{n!}{(n-r)!(r)!}\\ &= C^n_r\end{aligned}$$

3.
$$\begin{aligned}C^n_{r-1}+C^n_r &= \frac{n!}{(r-1)![n-(r-1)]!}+\frac{n!}{r!(n-r)!}\\ &= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{n-r+1}+\frac{1}{r}\right]\\ &= \frac{n!}{(r-1)!(n-r)!}\left[\frac{r+n-r+1}{(n-r+1)r}\right]\\ &= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!}\\ &= \frac{(n+1)!}{r!(n-r+1)!}\\ &= \frac{(n+1)!}{r![(n+1)-r]!}\\ &= C^{n+1}_r\end{aligned}$$

Appendix 3

Arithmetic and Geometric Series

Section 2.4 Sum to Infinity of a Geometric Series

KEY CONCEPTS AND FORMULAE

The sum to infinity of a geometric series $S(\infty)$ with the first term a and the common ratio R is given by

$$S(\infty) = \frac{a}{1-R},$$

where $-1 < R < 1$.

Example

Sum to infinite of the geometric series 18 + 12 + 8 + ……

Soln

$$\text{First term} = a = 18$$

$$\text{Common Ratio} = \frac{\text{second term}}{\text{first term}} = \frac{2}{3}$$

$$\therefore \quad S(\infty) = \frac{8}{\frac{2}{3}} = 12_{\#}$$