I. Exponential Function

A. Definition

$f(x) = a^x$, where ($a > 0$) and is the **base** and the independent variable x is the **exponent**.

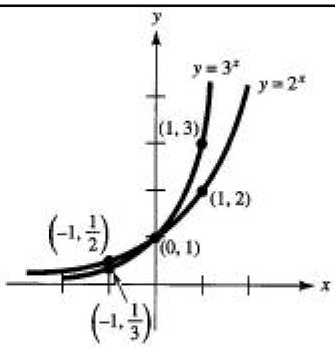
$[a^x = \underbrace{a \times a \times a \times \cdots \times a}_{x-times}]$

(***Remarks*** **:** $f(x) = x^n$ is a power function in which the *base* is the variable and the *exponent* is the constant)

B. Graphs

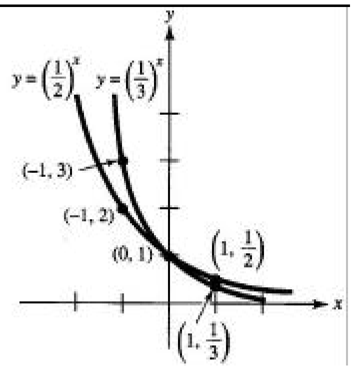
$a > 0$

x	2^x	3^x
-2	$\frac{1}{4}$	$\frac{1}{9}$
-1	$\frac{1}{2}$	$\frac{1}{3}$
0	1	1
1	2	3
2	4	9



$0 < a < 1$

x	$\left(\frac{1}{2}\right)^x$	$\left(\frac{1}{3}\right)^x$
-2	4	9
-1	2	3
0	1	1
1	$\frac{1}{2}$	$\frac{1}{3}$
2	$\frac{1}{4}$	$\frac{1}{9}$



Properties of Exponential Functions

a. The domain of an exponential function is all real numbers.
b. The range is all positive real numbers.
c. Each graph has a y-intercept (0, 1). There is no x-intercept.
d. The graph of $y = a^{-x}$ is the reflection, with respect to y-axis, of the graph of $y = a^x$.
e. the graph has two basic shapes, depending on whether $a > 1$ or $0 < a < 1$.

II. Laws of exponents (*Certificate Level*)

a. $a^m \times a^n = a^{m+n}$

b. $\dfrac{a^m}{a^n} = a^{m-n}$

c. $(a^m)^n = a^{mn}$

d. $(ab)^m = a^m b^m$

e. $(\dfrac{a}{b})^m = \dfrac{a^m}{b^m}$

f. $a^0 = 1$

g. $a^{-n} = \dfrac{1}{a^n}$

h. $a^{\frac{1}{p}} = \sqrt[p]{a}$

i. $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

III. Exponential Series

Consider the series $\left(1+\dfrac{x}{n}\right)^n$,

$$\left(1+\frac{x}{n}\right)^n = 1+n\left(\frac{x}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3+\cdots+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\left(\frac{x}{n}\right)^r+\cdots$$

$$= 1+x+\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{x^2}{2!}\right)+\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\left(\frac{x^3}{3!}\right)+\cdots+$$

$$\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)\left(\frac{x^r}{r!}\right)+\cdots$$

$$= 1+x+\left(1-\frac{1}{n}\right)\left(\frac{x^2}{2!}\right)+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(\frac{x^3}{3!}\right)+\cdots+$$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)\left(\frac{x^r}{r!}\right)+\cdots$$

When n tends to infinity,

$$\left(1+\frac{x}{n}\right)^n \to 1+x+\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\cdots+\left(\frac{x^r}{r!}\right)+\cdots$$

$$\therefore \quad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \boxed{\begin{aligned}&= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^r}{r!}+\cdots\\ &= e^x\end{aligned}}$$

or

$$= \sum_{r=0}^{\infty}\frac{x^r}{r!}$$

Example

Expand e^{-2x} and e^{x^2} in ascending powers of x as far as the term in x^4.

[**ANS:** $1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4-\cdots$ & $1+x^2+\frac{1}{2}x^4+\cdots$]

IV. Applications

A. Compound Interest

1. Compounded Annually

If interest rate is r % p.a., and compounds interest once per year, principal amount is P, then

At the end of 1st year,

$$P(1) = P + \underline{rP} = (1 + r)P$$

At the end of 2nd year,

$$P(2) = (1 + r)P + \underline{r(1 + r)P} = (1 + r)^2 P$$

Interest

At the end of 3rd year,

$$P(3) = (1 + r)^3 P$$

Then,

$\therefore$ **After n years, the amount on deposit is**

$$\boxed{P(n) = (1 + r)^n P}$$

How about the following cases ?

2. Compounded Monthly ?

3. Compounded Quarterly ?

4. Compounded Daily ?

2. Continuously compounded interest

If interest rate is r %, and compounds interest k times per year, principal amount is P, then

For interest is compounded quarterly ($k = 4$)

$$P(\frac{1}{4}) = (1+\frac{r}{4})\,P, \qquad P(\frac{2}{4}) = (1+\frac{r}{4})^2\,P, \qquad P(\frac{3}{4}) = (1+\frac{r}{4})^3\,P$$

At the end of 1st year,

$$P(1) = (1+\frac{r}{4})^4\,P$$

In general, if compounding k times per year at interest rate is r%, then

At the end of 1st year,

$$P(1) = (1+\frac{r}{k})^k\,P$$

At the end of 2nd year,

$$P(2) = (1+\frac{r}{k})^k\left[(1+\frac{r}{k})^k\,P\right]$$

$$= (1+\frac{r}{k})^{2k}\,P$$

∴ **The amount on deposit after n years is**

$$\boxed{\mathbf{P(n) = (1+\frac{r}{k})^{nk}\,P}}$$

Common values of k :	$k = 1$	annual
	$k = 4$	quarterly
	$k = 12$	monthly
	$k = 365$	daily

2. Continuously compounded interest *(cont.)*

For compounding continuously,

$$\begin{aligned} P(1) &= \lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{k} P \\ &= \lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{(k/r)r} P \\ &= \left[\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}\right]^{r} P \\ &= e^{r} P \end{aligned}$$

Amount on deposit after 2 years,

$$\begin{aligned} P(2) &= e^{r}[e^{r}P] \\ &= e^{2r}P \end{aligned}$$

$\therefore$ **Amount on deposit after n years,**

$$\boxed{\mathbf{P(n) = e^{rn} P}}$$

Exercise

A capital of \$60 000 is invested at a rate of 8% per annum for 5 years. Find the amount accumulated if interest is compounded

(a) yearly, ***[\$88 159.68]***
(b) monthly, ***[\$89 390.74]***
(c) continuously. ***[\$89 509.48]***

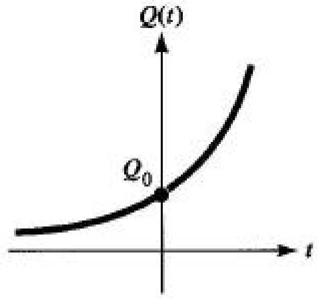
C. Exponential Growth & Decay

1. *Ordinary Differential Equation:*

2. *Exponential Models*

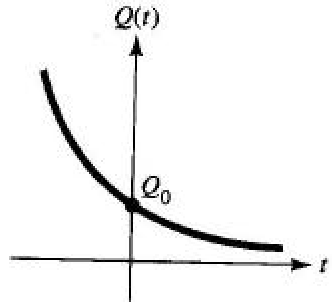
(i) Exponential Growth

$$\boxed{Q(t) = Q_o e^{kt}}$$



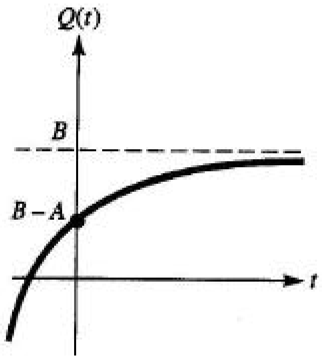
(ii) Exponential Decay

$$\boxed{Q(t) = Q_o e^{-kt}}$$



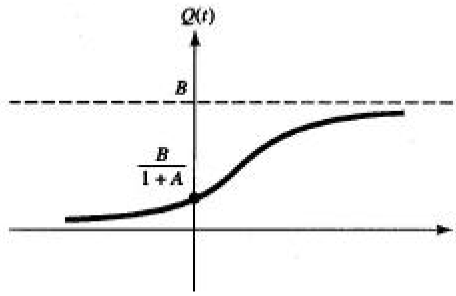
(iii) Learning Curves

$$\boxed{Q(t) = B - A e^{-kt}}$$



(iv) Logistic Curve

$$\boxed{Q(t) = \frac{B}{1 + Ae^{-Bkt}}}$$



3. *Examples*

(a) It is projected that t years from now, the population of a certain country will be $P(t) = 50\, e^{0.02t}$ million.

(i) What is the current population? [***50,000,000***]

(ii) What will the population be 30 years from now? [***91,105,940***]

(b) A certain industrial machine depreciates so that its value after t years is given by a function of the form $Q(t) = Q_o\, e^{-0.04t}$. After 20 years, the machine is worth \$8,986.58. What was its original value? [***20,000***]

(c) The rate at which a postal clerk can sort mail is a function of the clerk's experience. Suppose the postmaster of a large city estimates that after t months on the job, the average clerk can sort $Q(t) = 700 - 400\, e^{-0.5\,t}$ letters per hour?

(i) How many letters can a new employee sort per hour? [***300***]

(ii) How many letters can a clerk with 6 months' experience sort per hour? [***680***]

(d) Public health records indicate that t weeks after the outbreak of a certain form of influenza, approximately $Q(t) = \dfrac{20}{1+19e^{-1.2t}}$ thousand people had caught the disease.

(i) How many people had the disease when it first broke out? [***1***]

(ii) How many had caught the disease by the end of the second week? [***7.343***]

(iii) If the trend continues, approximately how many people in all will contract the disease? [***20***]