5.2 Limits of Functions

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I. *Definition of the Limit of a Function*

We say that a function $f(x)$ approaches a limit l as x tends to c, in symbols, $\lim_{x\to c} f(x)=l$

II. *Existence of the Limit*

If $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)$, then $\lim_{x\to c} f(x)$ exist.

5.3 Evaluation of Limits

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I. *Theorems on limits of functions*

a. $\lim_{x\to c} k = k$

b. $\lim_{x\to c}[kf(x)] = k\lim_{x\to c} f(x)$

c. $\lim_{x\to c}[f(x)\pm g(x)] = \lim_{x\to c} f(x)\pm \lim_{x\to c} g(x)$

d. $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x)\times \lim_{x\to c} g(x)$

e. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

f. $\lim_{x\to c}[f(x)]^k = [\lim_{x\to c} f(x)]^k$

g. $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$

II. *Limit of a Polynomial Function*

If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is a polynomial function, then

$$\lim_{x\to c} f(x)=a_nc^n+a_{n-1}c^{n-1}+\ldots+a_1c+a_0 \quad \Leftrightarrow \quad \textbf{Ia, Ib, Ic}$$

III. *Limit of an algebraic fraction*

a. Reducing the fraction to the lowest term, or Rationalizing the denominator

b. Apply Theorems on limits of functions

IV. *Evaluation of Limits* **(*different types of results*)**

a. Constant

$\lim_{x\to -6}\dfrac{3x+4}{5x-6}$ ***(Q9)***

$=\dfrac{\lim_{x\to -6}(3x+4)}{\lim_{x\to -6}(5x-6)}$ **Ie**

$=\dfrac{3\lim_{x\to -6}(x)+\lim_{x\to -6}(4)}{5\lim_{x\to -6}(x)-\lim_{x\to -6}(6)}$ **Ic, Ib**

$=\dfrac{3(-6)+4}{5(-6)-6}$ **Ia, II**

$=\dfrac{-14}{-36}$

$=\dfrac{7}{18}$

Exercise 5.3 Q1-11,33

b. Zero

$\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x+1}$ ***(Q26)***

$=\dfrac{\lim_{x\to 0}(\sqrt{x+1}-1)}{\lim_{x\to 0}(x+1)}$ **Ie**

$=\dfrac{\lim_{x\to 0}(\sqrt{x+1})-\lim_{x\to 0}(1)}{\lim_{x\to 0}(x)+\lim_{x\to 0}(1)}$ **Ic**

$=\dfrac{\sqrt{\lim_{x\to 0}(x+1)}-\lim_{x\to 0}(1)}{\lim_{x\to 0}(x)+\lim_{x\to 0}(1)}$ **If**

$=\dfrac{\sqrt{\lim_{x\to 0}(x)+\lim_{x\to 0}(1)}-\lim_{x\to 0}(1)}{\lim_{x\to 0}(x)+\lim_{x\to 0}(1)}$ **Ic**

$=\dfrac{\sqrt{0+1}-1}{0+1}$

$=\dfrac{1-1}{1}$ **Ia, II**

$=0$

c. Does not exist / undefine / ∞

$\lim_{x\to 1}\dfrac{3x+7}{(x-1)^2}$ ***(Q24)***

$=\dfrac{\lim_{x\to 1}(3x+7)}{\lim_{x\to 1}(x-1)^2}$ **Ie**

$=\dfrac{\lim_{x\to 1}(3x)+\lim_{x\to 1}(7)}{[\lim_{x\to 1}(x-1)]^2}$ **Ic, If**

$=\dfrac{3\lim_{x\to 1}(x)+\lim_{x\to 1}(7)}{[\lim_{x\to 1}(x)-\lim_{x\to 1}(1)]^2}$ **Ic, Ib**

$=\dfrac{3(1)+7}{(1-1)^2}$ **Ia, II**

$=\dfrac{10}{0}$

$\Rightarrow$ **does not exist**

Exercise 5.3 Q15,17,24,34

IV. *Evaluation of Limits* (***different types of results***)

d. Indeterminate form $\left(\frac{0}{0} or \frac{\infty}{\infty}\right)$

(i) ***checking:*** $\lim_{x\to -1}\frac{x^2-1}{x+1}=\frac{(-1)^2-1}{(-1)+1}=\frac{0}{0}\Rightarrow$ ***Indeterminate form***

$\lim_{x\to -1}\frac{x^2-1}{x+1}$ (***Q12***)

$=\lim_{x\to -1}\frac{(x+1)(x-1)}{x+1}$ ***Cancel common factors***

$=\lim_{x\to -1}(x-1)$

$=(-1)-1$ ***Apply theorems on Limits of Function***

$=-2$

Exercise 5.3 Q12-14,16,18-23,25,32

(ii) ***checking:*** $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\frac{\sqrt{0+1}-1}{0}=\frac{1-1}{0}=\frac{0}{0}\Rightarrow$ ***Indeterminate form***

$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$ (***Q26***)

$=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\times\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$ ***Rationalizing the denominator***

$=\lim_{x\to 0}\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}$

$=\lim_{x\to 0}\frac{(\sqrt{x+1})^2-1}{x(\sqrt{x+1}+1)}$

$=\lim_{x\to 0}\frac{x}{x(\sqrt{x+1}+1)}$

$=\lim_{x\to 0}\frac{1}{(\sqrt{x+1}+1)}$ ***Cancel common factors***

$=\frac{1}{(\sqrt{0+1}+1)}$ ***Apply theorems on Limits of Function***

$=\frac{1}{2}$

Exercise 5.3 Q15,17,24,34

(iii) ***checking:*** $\lim_{x\to\infty}\frac{x}{2x+1}=\frac{\infty}{2(\infty)+1}=\frac{\infty}{\infty}\Rightarrow$ ***Indeterminate form***

the above steps is used *for explanation only*, but poor in presentation. **(***used in Ch7***)**

$\lim_{x\to\infty}\frac{x}{2x+1}$

$=\lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{2x+1}{x}}$ ***Reducing the function to lowest form***

$=\lim_{x\to\infty}\frac{1}{2+\frac{1}{x}}$ (when x is very large, then $\frac{1}{x}\approx 0$)

$=\frac{1}{2+0}$ ***Apply theorems on Limits of Function***

$=\frac{1}{2}$

(iv) ***checking:*** $\lim_{x\to\infty}\frac{3x-11}{x^2-4x+3}=\lim_{x\to\infty}\frac{3x-11}{x^2-4x+3}=\frac{3(\infty)-11}{(\infty)^2-4(\infty)+3}=\frac{\infty}{\infty}\Rightarrow$ ***Indeterminate form***

the above steps is used *for explanation only*, but poor in presentation.

$$\lim_{x\to\infty}\frac{3x-11}{x^2-4x+3}$$

$$=\lim_{x\to\infty}\frac{\frac{3x-11}{x^2}}{\frac{x^2-4x+3}{x^2}}$$ ***Reducing the function to lowest form***

$$=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{11}{x^2}}{1-\frac{4}{x}+\frac{3}{x^2}}$$ (when x is very large, then $\frac{1}{x}\approx 0$)

$$=\frac{0-0}{1-0+0}$$ ***Apply theorems on Limits of Function***

$$=0$$

Exercise 5.3 Q26-31

Examples(Exercise 5.3)

4. $\lim_{x\to 1}(2x^3-3x^2+6x-7) = 2(1)^3-3(1)^2+6(1)-7=-2$

7. $\lim_{x\to e} x\ln x = e\ln e = e$

11. $\lim_{x\to 7}\sqrt[3]{2x^2-5x+1} = \sqrt[3]{2(7)^2-5(7)+1} = 4$

22. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

$$=\lim_{x\to 9}\frac{\sqrt{x}-3}{[(\sqrt{x})^2-3^2]}$$

$$=\lim_{x\to 9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$=\lim_{x\to 9}\frac{1}{(\sqrt{x}+3)}$$

$$=\frac{1}{(\sqrt{9}+3)}$$

$$=\frac{1}{6}$$

31. $\lim_{x\to 2}\frac{5x^2-20}{\sqrt{x^2+3}-\sqrt{7}}$

$$=\lim_{x\to 2}\frac{5x^2-20}{\sqrt{x^2+3}-\sqrt{7}}\cdot\frac{\sqrt{x^2+3}+\sqrt{7}}{\sqrt{x^2+3}+\sqrt{7}}$$

$$=\lim_{x\to 2}\frac{5(x^2-4)(\sqrt{x^2+3}+\sqrt{7})}{(x^2+3)-(7)}$$

$$=\lim_{x\to 2}\frac{5(x^2-4)(\sqrt{x^2+3}+\sqrt{7})}{x^2-4}$$

$$=\lim_{x\to 2}5(\sqrt{x^2+3}+\sqrt{7})$$

$$=5(\sqrt{2^2+3}+\sqrt{7})$$

$$=10\sqrt{7}$$

V. *Some Important Limits of function* (Ch2, P.62)

a. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

b. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

5.4 Derivatives of a Function (*first principles*) *Geometric interpretaton :- gradient of tangent* *P160*

$$\frac{d[f(x)]}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Examples(Exercise 5.4)

10.

$$y = 6x^2+5$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[6(x+h)^2+5]-(6x^2+5)}{h}$$

$$= \lim_{h\to 0}\frac{6(x^2+2hx+h^2)+5-6x^2-5}{h}$$

$$= \lim_{h\to 0}\frac{12hx+6h^2}{h}$$

$$= \lim_{h\to 0}(12x+6h)$$

$$= 12x$$

16.

$$y = \frac{1}{\sqrt{2x}}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\frac{1}{\sqrt{2(x+h)}}-\frac{1}{\sqrt{2x}}}{h}$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\cdot\frac{\sqrt{2x}-\sqrt{2(x+h)}}{\sqrt{2(x+h)}\cdot\sqrt{2x}}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\cdot\frac{\sqrt{2x}-\sqrt{2(x+h)}}{\sqrt{2(x+h)}\cdot\sqrt{2x}}\cdot\frac{\sqrt{2x}+\sqrt{2(x+h)}}{\sqrt{2x}+\sqrt{2(x+h)}}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\cdot\frac{2x-2(x+h)}{\sqrt{2(x+h)}\cdot\sqrt{2x}[\sqrt{2x}+\sqrt{2(x+h)}]}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2}{\sqrt{2(x+h)}\cdot\sqrt{2x}[\sqrt{2x}+\sqrt{2(x+h)}]}\right]$$

$$= \frac{-2}{\sqrt{2x}\cdot\sqrt{2x}\cdot 2\sqrt{2x}}$$

$$= \frac{-1}{(2x)^{\frac{3}{2}}}$$

17. (a)

$$y = x^3-x^2+6$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[(x+h)^3-(x+h)^2+6]-(x^3-x^2+6)}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^2-2xh-h^2+6-x^3+x^2-6}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-2xh-h^2}{h}$$

$$= \lim_{h\to 0}(3x^2+3xh+h^2-2x-h)$$

$$= 3x^2-2x$$

(b) **(i)** When $x = 3$, $\frac{dy}{dx} = 3(3)^2-2(3) = 21.$

(ii) When $x = -\frac{1}{2}$, $\frac{dy}{dx} = 3\left(-\frac{1}{2}\right)^2-2\left(-\frac{1}{2}\right) = \frac{7}{4}.$

(c) If $\frac{dy}{dx} = 1,$

then $3x^2-2x = 1$

$3x^2-2x-1 = 0$

$(3x+1)(x-1) = 0$

$x = -\frac{1}{3}$ or 1.