6.1 Differentiation of Sums & Powers of *x* — P.172

$\frac{d}{dx}(k)=0$ — *Constant rule*

$\frac{d}{dx}(x^n)=nx^{n-1}$ — *Power rule*

$\frac{d}{dx}[kf(x)]=k\frac{d}{dx}[f(x)]$ — *Constant multiple rule*

$\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$ — *Sum & Different rule*

6.2 Differentiation of Products & Quotients of functions — P.179

$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ — *Product rule*

$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ — *Quotient rule*

6.3 Differentiation of Composite functions — P.185

Suppose ***y = g(u)*** and ***u = h(x)***

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

6.4 Differentiation of Inverse functions — P.191

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

6.5 Differentiation of Implicit functions — P.197

$$\frac{d}{dx}(y^n)=ny^{n-1}\frac{dy}{dx}$$

6.6 Differentiation of Exponential functions — P.200

$$\frac{d}{dx}\left(e^x\right)=e^x$$

$$\frac{d}{dx}\left(e^{kx}\right)=ke^{kx}$$

6.7 Differentiation of Logarithmic functions — P.204

$$\frac{d}{dx}(\ln x)=\frac{1}{x}$$

6.8 Differentiation of $\log_a x$ & a^x — P.209

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$$

$$\frac{d}{dx}\left(a^x\right)=a^x\ln a$$

6.9 Second Derivatives — P.212

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Examples

EXERCISE 6.1

12. $f(x) = 3x^6 + 2x^3 - 7$

$f'(x) = 3\cdot 6(x^{6-1}) + 2\cdot 3(x^{3-1}) - 0$

$= 18x^5 + 6x^2$

23. $y = 3u^{\frac{1}{2}} - \dfrac{1}{5u^{\frac{1}{4}}}$

$= 3u^{\frac{1}{2}} - \dfrac{1}{5}u^{-\frac{1}{4}}$

$\dfrac{dy}{du} = 3\cdot\dfrac{1}{2}u^{\frac{1}{2}-1} - \dfrac{1}{5}\cdot(-\dfrac{1}{4})u^{-\frac{1}{4}-1}$

$= \dfrac{3}{2}u^{-\frac{1}{2}} + \dfrac{1}{20}u^{-\frac{5}{4}}$

30. $y = \dfrac{4\sqrt{t} - 3\sqrt[3]{t}}{6\sqrt[6]{t}} = \dfrac{4t^{\frac{1}{2}} - 3t^{\frac{1}{3}}}{6t^{\frac{1}{6}}}$

$= \dfrac{2}{3}t^{\frac{1}{3}} - \dfrac{1}{2}t^{\frac{1}{6}}$

$\dfrac{dy}{dt} = \dfrac{2}{3}\cdot\dfrac{1}{3}t^{\frac{1}{3}-1} - \dfrac{1}{2}\cdot\dfrac{1}{6}t^{\frac{1}{6}-1}$

$= \dfrac{2}{9}t^{-\frac{2}{3}} - \dfrac{1}{12}t^{-\frac{5}{6}}$

$\left.\dfrac{dy}{dt}\right|_{t=64} = \dfrac{2}{9}(64^{-\frac{2}{3}}) - \dfrac{1}{12}(64^{-\frac{5}{6}})$

$= \dfrac{13}{1152}$ or 0.0113

EXERCISE 6.2

1. $y = (3x - 2)(2x + 7)$

$\dfrac{dy}{dx} = (3x - 2)\dfrac{d}{dx}(2x + 7) + (2x + 7)\dfrac{d}{dx}(3x - 2)$

$= (3x - 2)(2) + (2x + 7)(3)$

$= 6x - 4 + 6x + 21$

$= 12x + 17$

4. $y = (3x^2 - 4)^2 = (3x^2 - 4)(3x^2 - 4)$

$y' = (3x^2 - 4)\dfrac{d}{dx}(3x^2 - 4) + (3x^2 - 4)\dfrac{d}{dx}(3x^2 - 4)$

$= (3x^2 - 4)(3\cdot 2x - 0) + (3x^2 - 4)(3\cdot 2x - 0)$

$= (3x^2 - 4)(6x) + (3x^2 - 4)(6x)$

$= 36x^3 - 48x$

14. $y = \dfrac{2x}{x + 1}$

$\dfrac{dy}{dx} = \dfrac{(x+1)\dfrac{d}{dx}(2x) - (2x)\dfrac{d}{dx}(x+1)}{(x+1)^2}$

$= \dfrac{(x+1)(2) - (2x)(1)}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2}$

26. $h(x) = \left(\dfrac{x-3}{x+1}\right)(2x - 5)$

$h'(x)$

$= \left(\dfrac{x-3}{x+1}\right)\dfrac{d}{dx}(2x-5) + (2x-5)\dfrac{d}{dx}\left(\dfrac{x-3}{x+1}\right)$

$= \left(\dfrac{x-3}{x+1}\right)(2) +$

$(2x-5)\cdot\dfrac{(x+1)\dfrac{d}{dx}(x-3) - (x-3)\dfrac{d}{dx}(x+1)}{(x+1)^2}$

$= \dfrac{2x-6}{x+1} + (2x-5)\cdot\dfrac{(x+1)(1) - (x-3)(1)}{(x+1)^2}$

$= \dfrac{(2x-6)(x+1) + (2x-5)(4)}{(x+1)^2}$

$= \dfrac{2x^2 + 4x - 26}{(x+1)^2}$

EXERCISE 6.3

1. $y = f(x) = (2x+5)^6$

i.e. $y = u^6$ where $u = 2x + 5$.

$\therefore$ $y = g(u) = u^6$

and $u = h(x) = 2x + 5$.

2. $y = f(x) = \sqrt[4]{x^2 + x + 1}$

i.e. $y = \sqrt[4]{u}$ where $u = x^2 + x + 1$.

$\therefore$ $y = g(u) = \sqrt[4]{u}$

and $u = h(x) = x^2 + x + 1$

4. $y = f(x) = e^{2x^2 - 9}$

i.e. $y = e^u$ where $u = 2x^2 - 9$

$\therefore$ $y = g(u) = e^u$

and $u = h(x) = 2x^2 - 9$.

8. $y = g(u) = (u+1)^3$, $u = h(x) = \dfrac{1}{x}$

$\dfrac{dy}{dx} = \dfrac{d}{du}[(u+1)^3]\cdot\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$

$= 3(u+1)^2\left(-\dfrac{1}{x^2}\right)$

$= -\dfrac{3}{x^2}\left(\dfrac{1}{x}+1\right)^2$

11. $y = (3x+4)^7$

$\dfrac{dy}{dx} = \dfrac{d(3x+4)^7}{d(3x+4)}\cdot\dfrac{d(3x+4)}{dx}$

$= 7(3x+4)^6(3)$

$= 21(3x+4)^6$

12. $y = (6x - 11)^{-3}$

$\dfrac{dy}{dx} = \dfrac{d(6x-11)^{-3}}{d(6x-11)}\cdot\dfrac{d(6x-11)}{dx}$

$= -3(6x-11)^{-4}(6)$

$= -18(6x-11)^{-4}$

25. $f(s) = \dfrac{(2s+1)^6}{s-1}$

$f'(s) = \dfrac{(s-1)\dfrac{d}{ds}(2s+1)^6 - (2s+1)^6\dfrac{d}{ds}(s-1)}{(s-1)^2}$

$= \dfrac{(s-1)\cdot 6(2s+1)^5(2) - (2s+1)^6(1)}{(s-1)^2}$

$= \dfrac{(2s+1)^5[12(s-1)-(2s+1)]}{(s-1)^2}$

$= \dfrac{(2s+1)^5(10s-13)}{(s-1)^2}$

31. $f(x) = (2x^2 + 3x - 1)^5$
$f'(x) = 5(2x^2 + 3x - 1)^4(4x + 3)$
$f'(-1) = 5[2(-1)^2 + 3(-1) - 1]^4[4(-1) + 3]$
$= -80$

EXERCISE 6.4

1. $y = 3x - 4$
$3x = y + 4$
$x = \frac{1}{3}(y + 4)$

5. $y = \sqrt{2x-1}$
$y^2 = 2x - 1$
$x = \frac{1}{2}(y^2 + 1)$

15. $x = 7y - 9$
$\dfrac{dx}{dy} = 7$
$\dfrac{dy}{dx} = \dfrac{1}{7}$

17. $x = (6 - y)^3$
$\dfrac{dx}{dy} = \dfrac{d(6-y)^3}{d(6-y)}\cdot\dfrac{d(6-y)}{dy}$
$\dfrac{dx}{dy} = 3(6-y)^2(-1)$
$\dfrac{dy}{dx} = \dfrac{-1}{3(6-y)^2}$ provided that $y \neq 6$.

EXERCISE 6.5

7. $2xy - 5 = 0$
$2[x\dfrac{dy}{dx} + y\dfrac{dx}{dx}] - 0 = 0$
$2[x\dfrac{dy}{dx} + y] = 0$
$\dfrac{dy}{dx} = -\dfrac{y}{x}$

12. $x^2 + 2xy + y^2 = 9$
$2x + 2[x\dfrac{dy}{dx} + y\dfrac{dx}{dx}] + 2y\dfrac{dy}{dx} = 0$
$2x + y + 2(x+y)\dfrac{dy}{dx} = 0$
$\dfrac{dy}{dx} = -\dfrac{2x+y}{2(x+y)}$

EXERCISE 6.6

5. $y = 4e^{-8x} + 15e^{\frac{x}{3}}$
$\dfrac{dy}{dx} = 4(-8e^{-8x}) + 15(\frac{1}{3}e^{\frac{x}{3}})$
$= -32e^{-8x} + 5e^{\frac{x}{3}}$

9. $f(x) = xe^x$
$f'(x) = x\dfrac{d}{dx}(e^x) + e^x\dfrac{d}{dx}(x)$
$= xe^x + e^x$

21. $g(t) = \dfrac{e^t + e^{-t}}{e^t - e^{-t}}$

$g'(t) = \dfrac{(e^t - e^{-t})\dfrac{d}{dt}(e^t + e^{-t}) - (e^t + e^{-t})\dfrac{d}{dt}(e^t - e^{-t})}{(e^t - e^{-t})^2}$

$= \dfrac{(e^t - e^{-t})(e^t - e^{-t}) - (e^t + e^{-t})(e^t + e^{-t})}{(e^t - e^{-t})^2}$

$= \dfrac{-4}{(e^t - e^{-t})^2}$

24. $h(r) = 4e^{(2r+5)^6}$

$h'(r) = 4\cdot\dfrac{de^{(2r+5)^6}}{d(2r+5)^6}\cdot\dfrac{d(2r+5)^6}{d(2r+5)}\cdot\dfrac{d(2r+5)}{dr}$

$= 4e^{(2r+5)^6}\cdot 6(2r+5)^5(2)$

$= 48(2r+5)^5 e^{(2r+5)^6}$

EXERCISE 6.7

3. $y = 4\ln(1 - 2x)$
$\dfrac{dy}{dx} = 4\cdot\dfrac{d\ln(1-2x)}{d(1-2x)}\cdot\dfrac{d(1-2x)}{dx}$
$= 4\cdot\dfrac{1}{1-2x}\cdot(-2)$
$= \dfrac{-8}{1-2x}$

16. $g(u) = \dfrac{\ln u^3}{u^2}$

$g'(u) = \dfrac{u^2\dfrac{d}{du}(\ln u^3) - (\ln u^3)\dfrac{d}{du}(u^2)}{u^4}$

$= \dfrac{u^2\left(\dfrac{1}{u^3}\cdot 3u^2\right) - (\ln u^3)(2u)}{u^4}$

$= \dfrac{3u - 2u\ln u^3}{u^4}$

$= \dfrac{3 - 2\ln u^3}{u^3}$

Logarithmic Differentiation

29. $y = (x^2 + 3)^4(2x^3 - 1)^5$
$\ln y = 4\ln(x^2 + 3) + 5\ln(2x^3 - 1)$
$\dfrac{1}{y}\dfrac{dy}{dx} = 4\cdot\dfrac{1}{x^2+3}\cdot(2x) + 5\cdot\dfrac{1}{2x^3-1}(6x^2)$
$\dfrac{dy}{dx} = [(x^2+3)^4(2x^3-1)^5]\left(\dfrac{8x}{x^2+3} + \dfrac{30x^2}{2x^3-1}\right)$
$= (x^2+3)^3(2x^3-1)^4[8x(2x^3-1) + 30x^2(x^2+3)]$
$= 2x(x^2+3)^3(2x^3-1)^4(23x^3 + 45x - 4)$

37.
$$y = x^x$$
$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}(x)$$
$$\frac{dy}{dx} = x^x\left(x \cdot \frac{1}{x} + \ln x\right)$$
$$= x^x(1 + \ln x)$$

EXERCISE 6.8

1.
$$y = \log_3 x$$
$$\frac{dy}{dx} = \frac{1}{x \ln 3}$$

3.
$$y = \log_4(5x + 7)$$
$$\frac{dy}{dx} = \frac{d\log_4(5x+7)}{d(5x+7)} \cdot \frac{d(5x+7)}{dx}$$
$$= \frac{1}{(5x+7)\ln 4} \cdot (5)$$
$$= \frac{5}{(5x+7)\ln 4}$$

17.
$$f(x) = x^7 5^x$$
$$f'(x) = x^7\frac{d}{dx}(5^x) + 5^x\frac{d}{dx}(x^7)$$
$$= x^7(5^x \ln 5) + 5x^x(7x^6)$$
$$= x^6 5^x(x \ln 5 + 7)$$

27.
$$h(x) = (\log_2 x)^x$$
$$\ln h(x) = x \ln(\log_2 x)$$
$$\frac{1}{h(x)} \cdot h'(x) = x\frac{d}{dx}[\ln(\log_2 x)] + [\ln(\log_2 x)]\frac{d}{dx}(x)$$
$$h'(x) = (\log_2 x)^x\left[x \cdot \frac{1}{\log_2 x} \cdot \frac{1}{x \ln 2} + \ln(\log_2 x)\right]$$
$$= (\log_2 x)^x\left[\frac{1}{(\log_2 x)(\ln 2)} + \ln(\log_2 x)\right]$$

EXERCISE 6.9

3.
$$y = 2x^3 - 10x^2 + 6x - 8$$
$$\frac{dy}{dx} = 6x^2 - 20x + 6$$
$$\frac{d^2y}{dx^2} = 12x - 20$$

18.
$$g(u) = \ln(1 - 8u)$$
$$g'(u) = \frac{1}{1-8u} \cdot (-8) = \frac{-8}{1-8u}$$
$$g''(u) = -8[(-1)(1-8u)^{-2}(-8)]$$
$$= -64(1-8u)^{-2}$$

28.
$$y = e^{kx}$$
$$\frac{dy}{dx} = k e^{kx}$$
$$\frac{d^2y}{dx^2} = k^2 e^{kx}$$
$$\because \quad \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$$
$$\therefore \quad k^2e^{kx} - 3(k e^{kx}) - 4(e^{kx}) = 0 \quad (e^{kx} \neq 0 \text{ for all } x)$$
$$k^2 - 3k - 4 = 0$$
$$(k-4)(k+1) = 0$$
$$k = -1 \text{ or } 4$$